

CONSTRUCTION OF PARADOXICAL DECOMPOSITIONS

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ABSTRACT. This paper is an application of a matrix combinatorial property named as normality, to construct a paradoxical decomposition for given non-amenable discrete group. We conjecture that every such group admits a system of equations with this property.

1. INTRODUCTION

In their paper Rosenblatt and Willis introduce a concept for groups to show that for an infinite discrete amenable group or a non-discrete amenable group G a net of positive, normalized functions in $L^1(G)$ can be constructed such that this net converges weak* to invariance but does not converge strongly to invariance [6]. This concept which is called configuration and a rather different form of that are also used to classify some group theoretical properties (see for example [1] and [2]). It is extended for hypergroups as well [9].

Configurations are strongly linked to the amenability of groups. On the other hand by Tarski's alternative a discrete group is non-amenable if and only if it admits a paradoxical decomposition. Therefore it is valuable to construct the paradoxical decomposition for such a group, using configurations. This problem which originally asked by Willis is answered partially in [5]. In that paper the paradoxical decomposition was constructed under a condition which cannot be stated independently.

In the present paper we pose a general matrix combinatorial conjecture under which the paradoxical decomposition is completely constructed. We also find a new upper bound for Tarski number of given non-amenable group.

Notations 1.1. The following notations are used throughout this paper

- \mathbb{N} , \mathbb{Z} and \mathbb{R} are the set of natural, integer and real numbers, respectively,
- \sqcup is the disjoint union of sets,
- $gA = \{ga; a \in A\}$, for a group G , $A \subseteq G$ and $g \in G$,
- $\mathcal{P}(X)$ is the power set of the set X ,
- $|X|$ is the the cardinal number of the set X ,
- A $(0, 1)$ -matrix is a matrix with entries in $\{0, 1\}$.

Key words and phrases. amenable group, configuration, paradoxical decomposition, Tarski number.

2. PRELIMINARIES

2.1. Matrix theory. Linear algebra and the theory of linear system of equations plays a fundamental role in modern Mathematics. Finding a solution i.e. an assignment of numbers to the variable such that the equations satisfied, is an important part of this theory.

In the following by a nontrivial, nonnegative and normalized solution to a homogenous system of linear equations with real coefficient, we mean respectively a solution including numbers which are not zero together, are all nonnegative and are all nonnegative with sum 1.

There is a significant amount of applicable theorems for a homogenous system of equations to have nontrivial nonnegative solutions. Gordan's theorem is a prominent one which has also applications in linear programming [4].

Theorem 2.1. *(Gordan 1873) Either a linear homogenous system of equations $AX = 0$ possesses a nontrivial solution in nonnegative variables or there exists an equation, formed by taking some linear combination of equations, that all positive coefficients. That is, either there exists an x such that*

$$Ax = 0, \quad 0 \neq x \geq 0$$

or there exists a vector m such that $m^t A > 0$ (has positive entries).

Remark 2.2. *It is easily verified that in Theorem 2.1, if A is a $(0, 1)$ -matrix, then the entries of m can be chosen in \mathbb{Z} .*

The main theorem of this paper is designed and proved under a conjecture (see Conjecture 2.5). To clarify the conjecture, we need some definitions.

Let $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ be a permutation for the set $\{1, \dots, n\}$. Then

$$P_\pi = \begin{pmatrix} e_{\pi(1)} \\ e_{\pi(2)} \\ \vdots \\ e_{\pi(n)} \end{pmatrix}$$

is called the permutation matrix associated to π , where e_i denotes the row vector of length n with 1 in the i -th position and 0 otherwise. When the permutation matrix P_π is multiplied with a matrix M from left, $P_\pi M$ will permute the rows of M by π . If A is a matrix with entries in $\{0, 1\}$, we say that A is a $(0, 1)$ -matrix.

If $P = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix}$ is a permutation matrix, by P^+ we mean the matrix with shifted rows, i.e.

$$P^+ = P_\rho P = \begin{pmatrix} P_2 \\ P_3 \\ \vdots \\ P_n \\ P_1 \end{pmatrix},$$

in which ρ is the cyclic permutation $(1 \ 2 \ \dots \ n)$. Throughout we use the notation

$$T = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \vdots & & & & \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} = \sum_{1 \leq j \leq i \leq n} E_{ij},$$

where E_{ij} is the matrix with 1 in ij position and 0 otherwise. When the matrix T is multiplied with a matrix M from left, j -th row of TM will be the sum of j first rows of M .

Definition 2.3. Let $\ell \in \mathbb{N}$ and $\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}$ and $\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix}$ be two $(0, 1)$ -matrices

with rows A_i, B_i . Let also the vector $\sum_{i=1}^n (B_i - A_i)$ has strictly positive entries. If there exists a permutation matrix P such that the matrix

$$(2.1) \quad TP(B - A) - P^+ A$$

has integer entries equal or greater than -1 , we say that the homogenous system of equations $(B - A)X = 0$ is normal.

Remark 2.4. It is clear that if $(B - A)X = 0$ is a normal system of equations, then this system has no non-zero non-negative solution.

Conversely if $(B - A)X = 0$ is a system of equations with no non-zero non-negative solution, then by Gordan's theorem there exists a vector $m = (m_1, \dots, m_n)$ such that $m^t(B - A)$ has strictly positive entries. If we permit $B - A$ to have repeated rows and insert the opposite of a row (exchanging the corresponding rows of A and B) if necessary, then m_i can be chosen in $\{0, 1\}$. Now omit $B_i - A_i$, where $m_i = 0$. Denote the modified matrix by $B - A$ again. So we can assume that $\sum_{i=1}^n (B_i - A_i)$ has strictly positive entries. I guess the following conjecture is true but I have not found an admissible solution to this, so far.

Conjecture 2.5. *Every system $(B - A)X = 0$ is normal, where A and B are two $(0, 1)$ -matrices in $\mathcal{M}_{n \times \ell}$ such that $\sum_{i=1}^n (B_i - A_i)$ has strictly positive entries.*

2.2. Non-amenable discrete groups. Let G be discrete group. Then G is called amenable if it admits a finitely additive probability measure μ on the σ -algebra $\mathcal{P}(G)$ such that

$$\mu(gA) = \mu(A), \quad (A \subseteq G, g \in G).$$

Definition 2.6. [8] *Let G be a group acting on a set X and suppose $E \subseteq X$. E is G -paradoxical (or, paradoxical with respect to G) if for some positive integers m, n there are pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m$ of E and $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that*

$$E = \bigsqcup_{i=1}^n g_i A_i = \bigsqcup_{j=1}^m h_j B_j.$$

A group G is called paradoxical if it is G -paradoxical, where G acts on itself by left multiplication. Clearly if G is a paradoxical group satisfying the above definition, then it cannot be amenable. Indeed if μ is a G -invariant probability measure, then

$$\begin{aligned} 1 &= \mu(G) = \sum \mu(A_i) + \sum \mu(B_j) \\ &= \sum \mu(g_i A_i) + \sum \mu(h_j B_j) = \mu(G) + \mu(G) = 2. \end{aligned}$$

In fact there is the following remarkable alternative due to Alfred Tarski.

Theorem 2.7. *Let G be a discrete group. Exactly one of the following happens*

- 1) G is paradoxical,
- 2) G is amenable.

There are different types of paradoxical decomposition. We draw the reader's attention to the next proposition

Proposition 2.8. [5, proposition 1.2] *Let G be a group. Then the following statements are equivalent*

- 1) *There exist a partition $\{A_1, \dots, A_n, B_1, \dots, B_m\}$ of G and g_1, \dots, g_n and h_1, \dots, h_m in G such that $\{g_i A_i\}_{i=1}^n$ and $\{h_j B_j\}_{j=1}^m$ form partitions of G .*
- 2) *There exist pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m$ of G and elements g_1, \dots, g_n and h_1, \dots, h_m in G such that $\{g_i A_i\}_{i=1}^n$ and $\{h_j B_j\}_{j=1}^m$ form partitions of G .*
- 3) *There exist pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m$ of G and elements g_1, \dots, g_n and h_1, \dots, h_m in G such that $G = \bigcup_{i=1}^n g_i A_i = \bigcup_{j=1}^m h_j B_j$ (not necessarily pairwise disjoint).*

Because of the above equivalence, we say that G admits a paradoxical decomposition, if any condition of Proposition 2.8 holds. However the decomposition in condition 1 is called a *complete* paradoxical decomposition.

The number $\tau = n+m$ for n and m in condition 1 of the previous definition is called the Tarski number of that paradoxical decomposition; the minimum of all such numbers over all the possible paradoxical decompositions of G , is called the Tarski number of G and denoted by $\tau(G)$. In the case that there is no paradoxical decomposition, we set $\tau(G) = \infty$. The Tarski number of a group is of real interest and has been estimated for some classes of groups. But it is not so easy to compute in many cases. However finding upper bounds for this number is not of low value to many authors and also to us. For rather detailed materials in the subject see [7] and [3].

2.3. Configuration of groups. Let G be discrete group. The configurations of G are defined in terms of finite generating sets and finite partitions of G .

If $\mathbf{g} = (g_1, \dots, g_n)$ is a string of elements of G and $\mathcal{E} = \{E_1, \dots, E_m\}$ is a partition of G , a configuration corresponding to $(\mathbf{g}, \mathcal{E})$ is an $(n+1)$ -tuple $C = (c_0, \dots, c_n)$, where $1 \leq c_i \leq m$ for each i , such that there is x in G with $x \in E_{c_0}$ and $g_i x \in E_{c_i}$ for each $1 \leq i \leq n$. The set of all configurations corresponding to the pair $(\mathbf{g}, \mathcal{E})$ will be denoted by $Con(\mathbf{g}, \mathcal{E})$. It is shown that groups with the same set of configurations have some common properties. For example they obey the same semigroup laws and have the same Tarski numbers (see [1] and [10]).

In the case that $\mathbf{g} = \{g_1, \dots, g_n\}$ is a generating set for G , the configuration $C = (c_0, \dots, c_n)$ may be described as a labelled tree which is a subgraph of the Cayley graph of the finitely generated group G and configuration set $Con(\mathbf{g}, \mathcal{E})$ is a set of rooted trees having height 1. In last section of the paper we assign a new graph to G that depends on the pair $(\mathbf{g}, \mathcal{E})$.

If $(\mathbf{g}, \mathcal{E})$ is as above and for each $C \in Con(\mathbf{g}, \mathcal{E})$

$$x_0(C) = E_{c_0} \cap (\cap_{j=1}^n g_j^{-1} E_{c_j}) \quad \text{and} \quad x_j(C) = g_j x_0(C),$$

then it is seen that for any $0 \leq j \leq n$ $\{x_j(C); C \in Con(\mathbf{g}, \mathcal{E})\}$ is a partition for G .

If D is a subset of $Con(\mathbf{g}, \mathcal{E})$, we use the following notation

$$\tilde{D} := \bigsqcup_{C \in D} x_0(C).$$

In particular $Con(\tilde{\mathbf{g}}, \mathcal{E}) = G$. To each pair $(\mathbf{g}, \mathcal{E})$ for G , there correspond a system of equations as follow

$$\sum_{x_j(C) \subseteq E_i} f_C = \sum_{x_k(C) \subseteq E_i} f_C, \quad (1 \leq i \leq m, 0 \leq j, k \leq n)$$

with variables f_C , $C \in Con(\mathbf{g}, \mathcal{E})$. This system is called the system of configuration equations corresponding to $(\mathbf{g}, \mathcal{E})$ and is denoted by $Eq(\mathbf{g}, \mathcal{E})$.

The existence of nonnegative solutions of this system plays an important role in amenability of groups and finding paradoxical decompositions, specially in this paper. This role was proved by Rosenblatt and Willis in the following theorem

Theorem 2.9. [6, Proposition 2.4] *There is a normalized solution of every possible instance of the configuration equations if and only if G is amenable.*

3. MAIN THEOREM

Throughout this section G is a group and $\mathbf{g} = (g_1, \dots, g_n)$ and $\mathcal{E} = \{E_1, \dots, E_m\}$ are a finite string of elements of G and a finite partition for G , respectively. The configuration equation $\sum_{x_j(C) \subseteq E_i} f_C = \sum_{x_k(C) \subseteq E_i} f_C$ is written in the form $aX = bX$, where $Con(\mathbf{g}, \mathcal{E}) = \{C_1, \dots, C_\ell\}$,

$$X = \begin{pmatrix} f_{C_1} \\ f_{C_2} \\ \vdots \\ f_{C_\ell} \end{pmatrix},$$

a is the coefficient vector of the left hand side and b is the coefficient vector of the right hand side of the equation.

Theorem 3.1. *If a subsystem of $Eq(\mathbf{g}, \mathcal{E})$ is normal, then G is non-amenable and a paradoxical decomposition of G can be written in terms of \mathbf{g}, \mathcal{E} .*

Proof. Let $V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix}, W = \begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{pmatrix} \in M_{n \times \ell}(\mathbb{R})$ with entries in $\{0, 1\}$

such that $(W - V)X = 0$ be the normal subsystem of $Eq(\mathbf{g}, \mathcal{E})$ satisfying (2.1). Hence the vector $(\alpha_C)_{C \in Con(\mathbf{g}, \mathcal{E})} := \sum_{i=1}^n (W_i - V_i)$ contains positive entries. Without loss of generality we can assume that $P = I$, the identity $n \times n$ -matrix. This means that there are strings (i_1, i_2, \dots, i_p) , $(j_{i_1}, j_{i_2}, \dots, j_{i_p})$ and $(k_{i_1}, k_{i_2}, \dots, k_{i_p})$ such that $i_s \in \{1, \dots, m\}$ and $j_{i_s}, k_{i_s} \in \{0, 1, \dots, n\}$ and the modified system is

$$(3.1) \quad \sum_{x_{j_{i_t}}(C) \subseteq E_{i_t}} f_C = \sum_{x_{k_{i_t}}(C) \subseteq E_{i_t}} f_C, \quad 1 \leq t \leq p.$$

Note that the strings are used instead of subsets, since the repetition is not excluded for the equations. For convenience for $1 \leq t \leq p$ we use the following notations

$$A_t = \{C; x_{k_{i_t}}(C) \subseteq E_{i_t}\} \text{ and } B_t = \{C; x_{j_{i_t}}(C) \subseteq E_{i_t}\}.$$

In other words, the system can be written as

$$\sum_{C \in A_t} f_C = \sum_{C \in B_t} f_C, \quad 1 \leq t \leq p.$$

This way the normality of (3.1) asserts that for each $m \leq n$

$$\bigcup_{k=1}^{m-1} (A_k \cap A_m) \subseteq \bigcup_{i=1}^{m-1} B_i.$$

Note that

$$g_{k_{i_t}}^{-1} g_{j_{i_t}} \bigcup_{C \in A_t} x_0(C) = \bigcup_{C \in B_t} x_0(C).$$

Set

$$\begin{aligned} S_1^1 &= g_1^{-1}(B_1 \setminus (A_1 \cap A_2))^{\sim}, \\ S_1^2 &= g_1^{-1}(A_1 \cap A_2)^{\sim}, \\ S_2^1 &= (A_2 \setminus A_1)^{\sim}. \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{B}_2 &= g_2 S_2^1 \sqcup g_2 g_1 S_1^2, \\ [B_1 \setminus (A_1 \cap A_2)]^{\sim} &= g_1 S_1^1. \end{aligned}$$

In the next step set

$$\begin{aligned} S_1^3 &= g_1^{-1}[B_1 \setminus (A_1 \cap A_2) \cap A_1 \cap A_3]^{\sim}, \\ S_1^4 &= g_1^{-1}[(B_1 \setminus (A_1 \cap A_2)) \cap A_2 \cap A_3]^{\sim}, \\ S_1^5 &= g_1^{-1}[(B_1 \setminus (A_1 \cap A_2)) \setminus (A_1 \cup A_2) \cap A_3]^{\sim}, \end{aligned}$$

and

$$\begin{aligned} S_2^2 &= g_2^{-1}[(B_2 \cap A_3 \cap A_1) \setminus (B_1 \setminus (A_1 \cap A_2)) \cap (A_1 \cup A_2) \cap A_3]^{\sim}, \\ S_2^3 &= g_2^{-1}[(B_2 \cap A_3 \cap A_2) \setminus (B_1 \setminus (A_1 \cap A_2)) \cap (A_1 \cup A_2) \cap A_3]^{\sim}, \\ S_2^4 &= g_2^{-1}[(B_2 \setminus (A_3 \cap (A_1 \cup A_2)))^{\sim}], \\ S_3^1 &= [A_3 \setminus (A_1 \cup A_2)]^{\sim} \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{B}_3 &= g_3 g_1 (S_1^3 \sqcup S_1^4) \sqcup g_3 g_2 g_1 [(g_1^{-1} S_2^2 \cap S_1^2) \sqcup (g_1^{-1} S_2^3 \cap S_1^2)] \\ &\quad \sqcup g_3 g_2 [(S_2^2 \cap S_2^1) \sqcup (S_2^3 \cap S_2^1)] \\ &\quad \sqcup g_3 S_3^1. \end{aligned}$$

Note that in the above formula we emphasis that

$$\begin{aligned} S_1^3, S_1^4, g_1^{-1} S_2^2 \cap S_1^2, g_1^{-1} S_2^3 \cap S_1^2 &\subseteq \tilde{A}_1, \\ S_2^2 \cap S_2^1, S_2^3 \cap S_2^1 &\subseteq (A_2 \setminus A_1)^{\sim}, \\ S_3^1 &\subseteq (A_3 \setminus (A_1 \cup A_2))^{\sim}. \end{aligned}$$

For $1 \leq k \leq m$ put

$$O_{k,m} = \begin{cases} \{(\sigma_k, \sigma_{k+1}, \dots, \sigma_m); \sigma_m = \sigma_k = 1, \sigma_i \in \{0, 1\}\} & \text{if } k \leq m-2 \\ \{(1, 1)\} & \text{if } k = m-1 \\ \{(1)\} & \text{if } k = m \end{cases}$$

Now for $\sigma = (\sigma_k, \dots, \sigma_m) \in O_{k,m}$ put

$$g_\sigma = \begin{cases} g_m^{\sigma_m} \dots g_k^{\sigma_k} & \text{if } k \leq m-2 \\ g_m g_k & \text{if } k = m-1 \\ g_k & \text{if } k = m \end{cases}$$

By an inductive process, for $1 \leq k \leq m$ and $\sigma \in O_{k,m}$ there is a subset \mathcal{A}_σ of P_k (probably empty set) such that

$$(3.2) \quad \tilde{Q}_m = \bigsqcup_{C \in Q_m} x_0(C) = \bigsqcup_{k=1}^m \bigsqcup_{\sigma \in O_{k,m}} (g_\sigma \mathcal{A}_\sigma)$$

Hence for $C \in Q_m$ we have

$$(3.3) \quad x_0(C) = \bigsqcup_{k=1}^m \bigsqcup_{\sigma \in O_{k,m}} (g_\sigma \mathcal{A}_\sigma \cap x_0(C)).$$

Note that $|O_{m,m}| = 1$ and $|O_{k,m}| = 2^{m-k-1}$ for $1 \leq k < m$. So $\sum_{k=1}^m |O_{k,m}| = 2^{m-1}$ disjoint subsets of $\bigsqcup P_k$ are used to construct \tilde{Q}_m .

Let

$$P'_i(C) := \begin{cases} 1 & C \in P_i \\ 0 & C \notin P_i \end{cases} \quad \text{and} \quad Q'_i(C) := \begin{cases} 1 & C \in Q_i \\ 0 & C \notin Q_i \end{cases}.$$

Then it can be easily seen that

$$(3.4) \quad \sum_{i=1}^n (W_i - V_i) = \sum_{i=1}^n (Q'_i - P'_i) = (\alpha_C)_{C \in \text{Con}(\mathfrak{g}, \mathcal{E})}$$

In addition P_1, P_2, \dots, P_n are pairwise disjoint. So for each $C \in \text{Con}(\mathfrak{g}, \mathcal{E})$

$$(3.5) \quad \sum_{i=1}^n P'_i(C) \in \{0, 1\}.$$

Therefore if for $C \in \text{Con}(\mathfrak{g}, \mathcal{E})$ we put $z_C = \alpha_C + \sum_{i=1}^n P'_i(C)$, there is a subset $\{m_1^C, m_2^C, \dots, m_{z_C}^C\}$ of $\{1, 2, \dots, n\}$ such that

$$C \in Q_{m_i^C}, \quad i = 1, \dots, z_C.$$

Therefore (3.3) can be viewed as a paradoxical decomposition for G . This decomposition is complete if for each $C \in \text{Con}(\mathfrak{g}, \mathcal{E})$,

$$\sum_{i=1}^n P'_i(C) = \alpha_C = 1.$$

Otherwise this decomposition is not complete but by a process described in the proof of [5, proposition 1.2] it can be changed into a complete one. \square

Regardless the group G is paradoxical or not, pursuing the notations in above proof, we have

$$\bigsqcup_{C \in B_i} x_0(C) = \tilde{B}_i = g_i \tilde{A}_i = g_i \bigsqcup_{C \in A_i} x_0(C).$$

It can be seen briefly in the following table

$A_1 \Rightarrow$	A_1			
$A_2 \Rightarrow$		A_2		
\vdots			\ddots	
$A_n \Rightarrow$				A_n
	\Downarrow B_1	\Downarrow B_2	...	\Downarrow B_n

In the case of non-amenability of G for every $C \in \text{Con}(\mathfrak{g}, \mathcal{E})$, $\alpha_C > 0$ and

$$|\{i; C \in A_i\}| + \alpha_C = |\{i; C \in B_i\}|.$$

Thus this is a paradoxical decomposition for the discrete group G (of any type) if and only if $\sum_{i=1}^n V_i(C) \in \{0, 1\}$, in other words, if A_1, A_2, \dots, A_n are pairwise disjoint. Since this does not happen in general case, we have constructed the following table with pairwise disjoint boxes, setting $\mathfrak{A}_{k,m} = \{\mathcal{A}_\sigma; \sigma \in O_{k,m}\}$.

$P_1 \Rightarrow$	\mathfrak{A}_{11}	\mathfrak{A}_{12}	\mathfrak{A}_{13}	...	$\mathfrak{A}_{1(n-1)}$	\mathfrak{A}_{1n}
$P_2 \Rightarrow$		\mathfrak{A}_{22}	\mathfrak{A}_{23}	...	$\mathfrak{A}_{2(n-1)}$	\mathfrak{A}_{2n}
$P_3 \Rightarrow$			\mathfrak{A}_{33}	...	$\mathfrak{A}_{3(n-1)}$	\mathfrak{A}_{3n}
\vdots					\vdots	\vdots
$P_{n-1} \Rightarrow$					$\mathfrak{A}_{(n-1)(n-1)}$	$\mathfrak{A}_{(n-1)n}$
$P_n \Rightarrow$						\mathfrak{A}_{nn}
	\Downarrow Q_1	\Downarrow Q_2	\Downarrow Q_3	...	\Downarrow Q_{n-1}	\Downarrow Q_n

Corollary 3.2. *Using the notations of the proof of Theorem 3.1, if for every $C \in \text{Con}(\mathfrak{g}, \mathcal{E})$, $\alpha_C = 1$, then $\tau(G) \leq (\ell - 1)(2^n - 1)$.*

Proof. By the explanation after (3.3) we have

$$\begin{aligned}
\tau(G) &\leq \sum_{m=1}^n (1 + \sum_{k=1}^{m-1} 2^{m-k-1}) |B_m| \\
&\leq \sum_{m=1}^n (1 + \sum_{k=1}^{m-1} 2^{m-k-1}) (\ell - 1) \\
&\leq \sum_{m=1}^n 2^{m-1} (\ell - 1) \\
&\leq (\ell - 1) (2^n - 1)
\end{aligned}$$

□

The bound found in Corollary 3.2 is not of course very accurate in many cases. But in the following we have an important bound for special groups.

4. GRAPH INTERPRETATION

In current section we assign a graph to a group which helps us to construct the paradoxical decompositions and to compute the Tarski numbers.

Definition 4.1. Let G be a group and $\mathbf{g} = (g_1, \dots, g_n)$ and $\mathcal{E} = \{E_1, \dots, E_m\}$ be a finite string of elements of G and a finite partition for G , respectively. $\Gamma = \Gamma(G, \mathbf{g}, \mathcal{E})$ is a graph constructed as follows

- The vertex set of Γ is identified with

$$V(\Gamma) := \{(a_C)_{C \in \text{Con}(\mathbf{g}, \mathcal{E})}; \ a_C \in \mathbb{N} \cup \{0\}\}.$$

- There exists a directed edge from the vertices $A = (a_C)$ to $B = (b_C)$ if there is a partition $\{A_1, \dots, A_b\}$ for

$$\bigsqcup_{a_C \neq 0} x_0(C)$$

such that for all $C \in \text{Con}(\mathbf{g}, \mathcal{E})$ there are $i_1, \dots, i_{b_C} \in \{1, \dots, b\}$ and $g_{i_1}, \dots, g_{i_{b_C}} \in G$, such that

$$x_0(C) = g_j A_j, \quad 1 \leq j \leq b_C,$$

provided that $b_C \neq 0$, where $b = \sum_{C \in \text{Con}(\mathbf{g}, \mathcal{E})} b_C$.

Proposition 4.2. If Γ contains adjacent vertices $A = (a_C)_{C \in \text{Con}(\mathbf{g}, \mathcal{E})}$ and $B = (b_C)_{C \in \text{Con}(\mathbf{g}, \mathcal{E})}$ with $a_C \in \{0, 1\}$ and $\alpha_C := b_C - a_C > 0$, then G admits a paradoxical decomposition in terms of \mathbf{g} and \mathcal{E} .

Proof. At first suppose that for each C $a_C = \alpha_C = 1$. In this case by assumption there is a partition $\{A_C; \ C \in \text{Con}(\mathbf{g}, \mathcal{E})\} \sqcup \{A'_C; \ C \in \text{Con}(\mathbf{g}, \mathcal{E})\}$ for $\cup_{a_C \neq 0} x_0(C) = G$ such that for each $C \in \text{Con}(\mathbf{g}, \mathcal{E})$

$$x_0(C) = g_C A_C = g'_C A'_C.$$

Therefore

$$G = \bigsqcup_{C \in \text{Con}(\mathfrak{g}, \mathcal{E})} x_0(C)$$

$$G = \left(\bigsqcup_{C \in \text{Con}(\mathfrak{g}, \mathcal{E})} \{A_C; C \in \text{Con}(\mathfrak{g}, \mathcal{E})\} \right) \sqcup \left(\bigsqcup_{C \in \text{Con}(\mathfrak{g}, \mathcal{E})} \{A'_C; C \in \text{Con}(\mathfrak{g}, \mathcal{E})\} \right)$$

and

$$G = \bigsqcup_{C \in \text{Con}(\mathfrak{g}, \mathcal{E})} g_C A_C = \bigsqcup_{C \in \text{Con}(\mathfrak{g}, \mathcal{E})} g'_C A'_C$$

which is clearly a complete paradoxical decomposition for G .

Now if there exists a $C \in \text{Con}(\mathfrak{g}, \mathcal{E})$ such that $\alpha_C = 0$, then there is a partition $\{A_C; C \in \text{Con}(\mathfrak{g}, \mathcal{E})\} \sqcup \{A'_C; \alpha_C \neq 0\}$ for $\cup_{\alpha_C \neq 0} x_0(C) = G$ such that for each $C \in \text{Con}(\mathfrak{g}, \mathcal{E})$

$$x_0(C) = g'_C A'_C$$

and

$$x_0(C) = g_C A_C, \quad (\alpha_C \neq 0).$$

Clearly

$$\text{Con}(\mathfrak{g}, \mathcal{E}) = \{C \in \text{Con}(\mathfrak{g}, \mathcal{E}); \alpha_C = 0\} \sqcup \{C \in \text{Con}(\mathfrak{g}, \mathcal{E}); \alpha_C \neq 0\}.$$

So

$$\begin{aligned} G &= \bigsqcup_{\alpha_C=0} x_0(C) \sqcup \left(\bigsqcup_{\alpha_C \neq 0} \{A_C; C \in \text{Con}(\mathfrak{g}, \mathcal{E})\} \right) \\ &\sqcup \left(\bigsqcup_{C \in \text{Con}(\mathfrak{g}, \mathcal{E})} \{A'_C; C \in \text{Con}(\mathfrak{g}, \mathcal{E})\} \right). \end{aligned}$$

Therefore

$$\begin{aligned} G &= \bigsqcup_{C \in \text{Con}(\mathfrak{g}, \mathcal{E})} x_0(C) \\ &= \left(\bigsqcup_{\alpha_C=0} x_0(C) \right) \sqcup \left(\bigsqcup_{\alpha_C \neq 0} x_0(C) \right) \\ &= \left(\bigsqcup_{\alpha_C=0} x_0(C) \right) \sqcup \left(\bigsqcup_{\alpha_C \neq 0} g_C A_C \right) = \bigsqcup_{C \in \text{Con}(\mathfrak{g}, \mathcal{E})} g'_C A'_C \end{aligned}$$

and

$$\begin{aligned} G &= \left(\bigsqcup_{C \in \text{Con}(\mathfrak{g}, \mathcal{E})} \{A_C; C \in \text{Con}(\mathfrak{g}, \mathcal{E})\} \right) \sqcup \left(\bigsqcup_{\alpha_C \neq 0} \{A'_C; C \in \text{Con}(\mathfrak{g}, \mathcal{E})\} \right) \\ &\sqcup \left(\bigsqcup_{\alpha_C=0} x_0(C) \right) \end{aligned}$$

is a paradoxical decomposition. The general case is similar. \square

Theorem 4.3. *Let G be a group and $\mathbf{g} = (g_1, \dots, g_n)$ and $\mathcal{E} = \{E_1, \dots, E_m\}$ be a finite string of elements of G and a finite partition for G , respectively. If $Eq(\mathbf{g}, \mathcal{E})$ has no nonnegative nonzero solution, then $\Gamma = \Gamma(G, \mathbf{g}, \mathcal{E})$ includes vertices $A = (a_C)_{C \in \text{Con}(\mathbf{g}, \mathcal{E})}$ and $B = (b_C)_{C \in \text{Con}(\mathbf{g}, \mathcal{E})}$ with $a_C \in \{0, 1\}$ and $\alpha_C := b_C - a_C > 0$.*

Proof. Using notations of the proof of Theorem 3.1, $A = \sum P'_i$ and $B = \sum Q'_i$, are desired vertices. \square

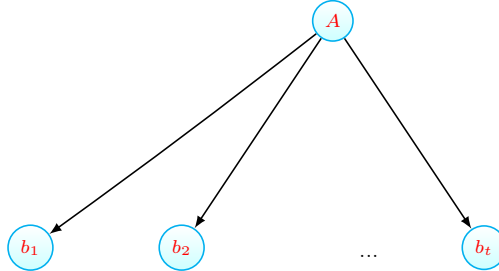
Let $A = \{a_1, \dots, a_s\}$ and $B = \{b_1, \dots, b_t\}$ be two subset of $\text{Con}(\mathbf{g}, \mathcal{E})$. If there is $g \in G$ and there is a partition $\{A_1, \dots, A_t\}$ for

$$\bigsqcup_{i=1}^s x_0(a_i)$$

such that

$$x_0(b_j) = gA_j, \quad 1 \leq j \leq t,$$

we use the following diagram



The composition of these diagrams leads us to construct the paradoxical decomposition described in Theorem 3.1. As the paradoxical decompositions of a group is not unique, different diagrams are also exist. One can find a Tasrki number's upper bound by counting the paths from the beginning to the ending points in the minimal diagram.

Example 4.4. *Let $C_1 = (1, 2, 3, 2)$, $C_2 = (1, 3, 1, 3)$, $C_3 = (2, 1, 2, 2)$, $C_4 = (3, 3, 1, 2)$, $C_5 = (3, 3, 2, 1)$ be the set of configurations of a group G . Then the configuration equation system of these configurations is*

$$\begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} f_{C_1} \\ f_{C_2} \\ f_{C_3} \\ f_{C_4} \\ f_{C_5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Considering any possible equation in 3.1, regardless the order of them, we add the following three systems to above

$$\begin{pmatrix} 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{C_1} \\ f_{C_2} \\ f_{C_3} \\ f_{C_4} \\ f_{C_5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

and

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_{C_1} \\ f_{C_2} \\ f_{C_3} \\ f_{C_4} \\ f_{C_5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

and

$$\begin{pmatrix} 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{C_1} \\ f_{C_2} \\ f_{C_3} \\ f_{C_4} \\ f_{C_5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

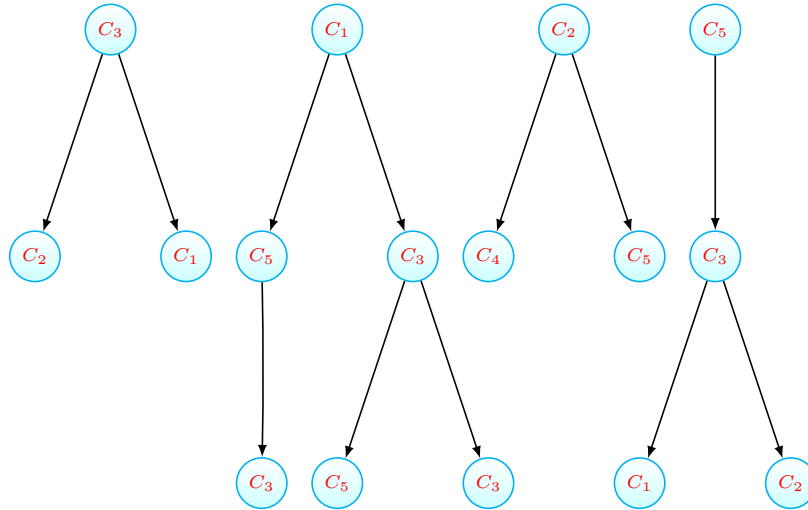
A modified subsystem corresponded to this system is $(B - A)X = 0$, where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

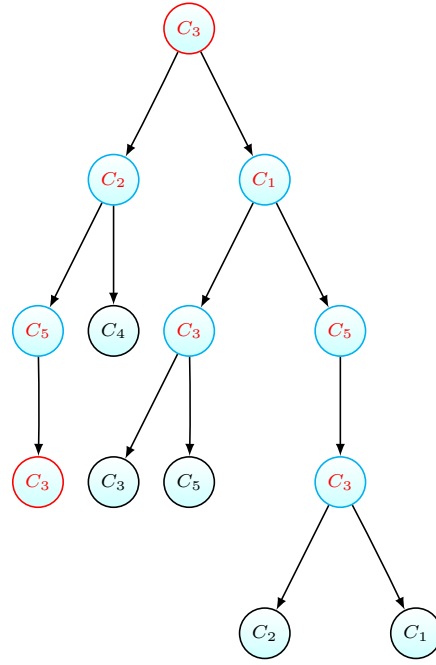
Setting $\pi = (2 \ 7)(4 \ 6)$, one have

$$TP_\pi(B - A) - P_\pi^+ A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Therefore this system is normal. The diagram of this system is



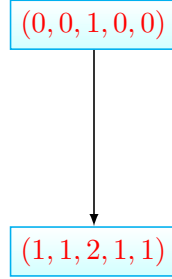
or equivalently



Since the in the end points of the diagram one can see the beginning points two times and the other configurations in $\text{Con}(\mathfrak{g}, \mathcal{E})$ at least once, this diagram turns to a paradoxical decomposition for G .

The number of paths from beginning to the end is 6. Therefore $\tau(G) \leq 6 + 1 = 7$.

The suitable part of $\Gamma(G, \mathfrak{g}, \mathcal{E})$ corresponding to this example is



We have a very special case of Theorem 3.1 which leads us to a result to compute the Tarski number. We use the notations of section 3.

Theorem 4.5. Suppose that $\sum_{i=1}^n (B_i - A_i) = (1, 1, \dots, 1)$ and there exists a permutation matrix P such that the first $n - 1$ rows of $PB - P^+A$ has nonnegative entries. Then $\tau(G) \leq 1 + |A_{P(1)}| + \ell$.

Proof. The assumption says that for $1 \leq i \leq n$, $V_{P(i+1)} \subseteq W_{P(i)}$. Let $V_{P(n+1)} = \emptyset$, $h_i = g_1^{-1} \dots g_i^{-1}$ and $R_i = h_i(W_i \setminus V_{i+1})$, for $1 \leq i \leq n$.

By induction on n we see

$$V_1 = \bigsqcup_{i=1}^n R_i.$$

Therefore the equations

$$\bigsqcup_{C \in (W_{P(i-1)} \setminus V_{P(i)})} x_0(C) = h_i^{-1} R_i, \quad (1 \leq i \leq n)$$

introduce a paradoxical decomposition. On the other hand we have

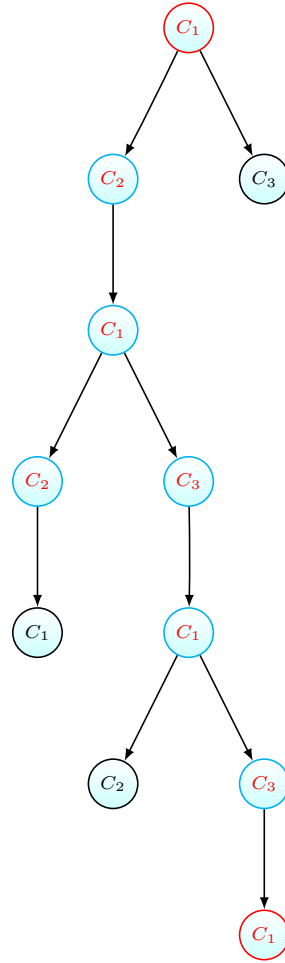
$$\begin{aligned} \tau(G) &\leq |W_{P(n)}| + \left(\sum_{i=2}^n |W_{P(i-1)} \setminus V_{P(i)}| \right) + 1 \\ &= \left(\sum_{i=1}^n |W_{P(i)} - V_{P(i)}| \right) + |V_{P(1)}| + 1 \\ &= \ell + |V_{P(1)}| + 1. \end{aligned}$$

□

In the special case where G does not contain the free group on two generators and $\ell = 3$ and $|V_1| = 1$ we have $\tau(G) = 5$.

Example 4.6. Let $Con(\mathfrak{g}, \mathcal{E}) = \{(1, 2, 2, 2), (2, 1, 2, 1), (2, 2, 1, 1)\}$ and G does not contain non-abelian free groups. Then $\tau(G) = 5$.

In fact since G does not contain non-abelian free groups, $\tau(G) > 4$ (see [3]). On the other hand a minimal diagram associated to $Con(\mathfrak{g}, \mathcal{E})$ is



The number of paths from the beginning to the end is 4. Therefore $4 < \tau(G) \leq 4 + 1 = 5$. So $\tau(G) = 5$.

REFERENCES

- [1] A. Abdollahi, A. Rejali and G. A. Willis, *Group properties characterized by configuration*, Illinois J. Mathematics, **48** (2004) No. 3, 861-873.
- [2] A. Abdollahi, A. Rejali and A. Yousofzadeh, *Configuration of nilpotent groups and isomorphism*, Journal of algebra and its applications, Vol. 8, No. 3 (2009) 339-350.
- [3] T. G. Ceccherini-Silberstein, R. I. Grigorchuk and P. de la Harpe, *Amenability and paradoxical decompositions for pseudogroups and for discrete metric spaces*, Proc. Steklov. Inst. Math. 224 (1999), 57-97.
- [4] G. B. Dantzig and M. N. Thapa, *Linear Programming 2: Theory and Extensions* (Springer-Verlag, New York, 2003).
- [5] A. Rejali and A. Yousofzadeh, *Configuration of groups and paradoxical decompositions*, Bull. Belg. Math. Soc. Simon Stevin 18 (2011) 157-172.
- [6] J. M. Rosenblatt and G. A. Willis, *Weak convergence is not strong for amenable groups*, Canadian Mathematical Bulletin, **44** (2)(2001), 231-241.
- [7] M. Sapir, *Combinatorial algebra: Syntax and Semantics* (Springer International Publishing, Switzerland, 2014).

- [8] S. Wagon, The Banach-Tarski Paradox, vol. 24 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge-New York, 1985.
- [9] B. Willson, Configurations and invariant nets for amenable hypergroups and related algebras, Trans. Amer. Math. Soc. 366 (2014), no. 10, 50875112. MR 3240918
- [10] A. Yousofzadeh, A. Tavakoli and A. Rejali, On configuration graph and paradoxical decomposition, Journal of algebra and its applications, Vol. 13, No. 2 (2014), 1350086 (11 pages).